MATH 426/529: Extremal Combinatorics

Term Test 2

November 23, 2026

Name: Solutions

Student number: _____

Instructions.

- This exam is 80 minutes.
- There are 36 marks available.
- No collaboration is allowed.
- Write complete solutions. Unless otherwise stated, justify your answers.

Question	Marks available	Marks earned
1	8	
2	6	
3	10	
4	12	
Total	36	

Question 1. [8 marks]

(a) Define the *extremal number* $\mathrm{ex}(n, H)$ of a graph H.

(b) Define a *graph homomorphism* from a graph H to a graph G.

(c) Let G be a graph and let $\mathcal{P} = (X_1, X_2, \ldots, X_k)$ be a partition of $V(G)$. Write down the expression for the *mean square density* of $\mathcal{P}$.

(d) Let G be a graph with n vertices and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the adjacency matrix of G. For $k \geq 3$, write down an expression for $t(C_k, G)$ in terms of $n, \lambda_1, \ldots, \lambda_n$. You do not need to prove it.

Write your answer to Question 1 here

(a) $\mathrm{ex}(n,H)$ is the maximum number of edges in an n-vertex graph that does not contain a copy of H.

(b) A function $f: V(H) \to V(G)$ is a graph homomorphism from H to G if, for every $uv \in E(H)$, it holds that $f(u)f(v) \in E(G)$.

(c) The mean square density of $\mathcal{P}$ is

$$\frac{1}{n^2} \sum_{\substack{(i,j) \in [k]^2 \\ (i \neq j)}} |X_i||X_j| d(X_i, X_j)^2.$$

(d) $$t(C_k, G) = \sum_{i=1}^{n} \left(\frac{\lambda_i}{n}\right)^k.$$

Continue your answer to Question 1 here

Continue your answer to Question 1 here

Continue your answer to Question 1 here

Question 2. [6 marks]

State each of the following theorems from lectures.

(a) Mantel's Theorem.

(b) The Erdős–Stone Theorem.

(c) The Triangle Removal Lemma.

Write your answer to Question 2 here

(a) If G is an n-vertex triangle-free graph, then $|E(G)| \leq \frac{n^2}{4}$. In other words, $ex(n, K_3) \leq \frac{n^2}{4}$.

(b) For any graph H with at least one edge,

$$ex(n, H) = \left(\frac{\chi(H)-2}{\chi(H)-1} + o(1)\right)\left(\frac{n^2}{2}\right).$$

(c) For every $0 < \varepsilon < \frac{1}{2}$, there exists $\delta > 0$ such that if G is a graph with at most $\delta |V(G)|^3$ triangles, then it can be made triangle-free by deleting at most $\varepsilon |V(G)|^2$ edges.

Continue your answer to Question 2 here

Continue your answer to Question 2 here

Continue your answer to Question 2 here

Question 3. **[10 marks]**

(a) Let $t \geq 2$, and let G be a $K_{2,t}$-free graph on n vertices. Prove that

$$\sum_{v \in V(G)} \binom{d(v)}{2} \leq (t-1)\binom{n}{2}.$$

(b) Let $k \geq 2$. Prove that every graph G satisfies $t(K_{1,k}, G) \geq t(K_2, G)^k$.

Write your answer to Question 3 here

(a) Count the pairs (v, S) where $v \in V(G)$ and $S \subseteq N(v)$ with $|S| = 2$. The number of such sets S for a given vertex v is $\binom{d(v)}{2}$. So, the total number of such pairs is

$$\sum_{v \in V(G)} \binom{d(v)}{2}.$$

On the other hand, a set $S \subseteq V(G)$ with $|S| = 2$ cannot be contained in more than $t-1$ such pairs, or else it would give us a copy of $K_{2,t}$ in G. Thus, the number of such pairs is $\leq (t-1)\binom{n}{2}$. So,

$$\sum_{v \in V(G)} \binom{d(v)}{2} \leq (t-1)\binom{n}{2}.$$

Continue your answer to Question 3 here

(b) Let G be a graph with n vertices.

Then

$$\hom(K_{1,k}, G) = \sum_{v \in V(G)} d(v)^k$$

$$\geq \frac{1}{n^{k-1}} \left(\sum_{v \in V(G)} d(v) \right)^k \quad \text{by convexity of } x^k.$$

$$= \frac{1}{n^{k-1}} \left(2|E(G)| \right)^k \quad \text{by handshaking}$$

Therefore, $t(K_{1,k}, G) = \dfrac{\hom(K_{1,k}, G)}{n^{k+1}} \geq \dfrac{(2|E(G)|)^k}{n^{2k}}$.

Also, $\hom(K_2, G) = 2|E(G)|$

and so $t(K_2, G) = \dfrac{2|E(G)|}{n^2}$.

Thus, ~~$t(K_{1,k}) \geq t(K_2, G)$~~

$$t(K_{1,k}, G) \geq t(K_2, G)^k.$$

□

Continue your answer to Question 3 here

Continue your answer to Question 3 here

Question 4. [12 marks]

Let $0 < \varepsilon < 1/3$. Let A, B, C be three disjoint non-empty vertex sets in a graph G. Suppose that each of the pairs

$$(A,B), \qquad (A,C), \qquad (B,C)$$

is ε-regular and has density at least 3ε.

(a) Prove that all but at most $\varepsilon|A|$ vertices $a \in A$ have at least

$$2\varepsilon|B|$$

neighbours in B.

(b) Prove that there is a vertex $a \in A$ which has at least $2\varepsilon|B|$ neighbours in B and at least $2\varepsilon|C|$ neighbours in C.

(c) Prove that G contains a triangle with one vertex in each of A, B, C. Prove this directly without using any theorems, lemmas, etc, proved in class.

Write your answer to Question 4 here

(a) Let A' be the set of all $a \in A$ with fewer than $2\varepsilon|B|$ neighbours in B. Suppose, to the contrary, that $|A'| \geq \varepsilon|A|$. Then $e(A',B) = \sum_{a \in A'} |N(a) \cap B| < 2\varepsilon|A'||B|$.

Therefore $d(A',B) < 2\varepsilon$ and so

$$|d(A',B) - d(A,B)| \geq d(A,B) - d(A',B) > 3\varepsilon - 2\varepsilon = \varepsilon.$$

However, since $|A| \geq \varepsilon|A|$ and $|B| > \varepsilon|B|$, this contradicts the fact that (A,B) is ε-regular.

(b) By (a), there are at most $\varepsilon|A|$ vertices of A with less than $2\varepsilon|B|$ neighbours in B. Similarly, there are at most $\varepsilon|A|$ vertices of A with less than $2\varepsilon|C|$ neighbours in C. So, there are at least $|A| - 2\varepsilon|A| = (1-2\varepsilon)|A| > 0$ vertices with at least $2\varepsilon|B|$ neighbours in B and at least $2\varepsilon|C|$ in C.

Continue your answer to Question 4 here

(c) Let $a \in A$ such that a has $2\varepsilon|B|$ neighbours in B and at least $2\varepsilon|C|$ neighbours in C. Let $X = N(v) \cap B$ and $Y = N(v) \cap C$. Then $|X| \geq \varepsilon|B|$ and $|Y| \geq \varepsilon|C|$.

Since (B,C) is ε-regular, we have

$$d(X,Y) \geq d(B,C) - \varepsilon \geq 3\varepsilon - \varepsilon = 2\varepsilon > 0.$$

Therefore $e(X,Y) > 0$. Let bc be an edge such that $b \in X$ and $c \in Y$. Then abc is a triangle. □

Continue your answer to Question 4 here

Continue your answer to Question 4 here